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LETTER TO THE EDITOR

Failure of hyperscaling in one-dimensional long-range bond percolation

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Abstract. We introduce a result which allows one to deduce the cluster number exponent $2 - \alpha$ in long-range one-dimensional percolation problems. This is used to show that in the long-range bond problem hyperscaling fails when the bond length exceeds 6.

One-dimensional percolation is one of the few systems in which one can analytically test scaling relations and notions of exponent universality. In higher dimensions the evidence for the current picture of exponent relations in percolation (see for example the reviews by Stauffer (1979) and Essam (1980)) is largely numerical. Although this evidence is generally convincing, it is nonetheless instructive to find simple systems in which scaling concepts may be checked precisely.

In long-range one-dimensional site percolation it was found (Klein *et al* 1978) that the generating function for the mean number of clusters on I^L , the one-dimensional lattice with connections up to L neighbours, was asymptotically a generalised homogeneous function of the site vacancy density $q = 1 - p$, and the ghost site density h . Although the scaling power of q was found to be L -dependent, the hyperscaling relation $\nu = 2 - \alpha$, connecting the correlation length exponent ν to the cluster number exponent α , held for all L . Furthermore the renormalised exponents of Suzuki (1974), namely

$$\hat{\gamma} = \gamma/\nu \quad \hat{\beta} = \beta/\nu \quad \hat{\phi} = (2 - \alpha)/\nu \quad (1)$$

where γ and β correspond to the mean cluster size and percolation probability singularities respectively, were found to be range-independent corresponding to 'weak' universality. Recently, the range dependence of the usual exponents has been shown to be a direct consequence of the triviality of the percolation threshold ($q_c = 0$) and thus established as a one-dimensional characteristic (Ord and Whittington 1985).

The bond problem on I^L has not been solved analytically for $L > 1$. However work by Zhang *et al* (1983) has shown that the correlation length exponent is given by $\nu(L) = L(L+1)/2$. By analogy with the site problem one might expect that the cluster number exponent is given by $2 - \alpha(L) = L(L+1)/2$, and that the renormalised exponents given in (1) are range-independent. In this letter we shall show that this is only the case for $L \leq 6$.

To define the exponent α one usually writes (e.g. Stauffer 1979)

$$K_s(p) \sim |p - p_c|^{2-\alpha}$$

or

$$\lim_{p \rightarrow p_c^-} \log K_s(p) / \log |p - p_c| = 2 - \alpha \quad (2)$$

where the subscript s refers to the 'singular part' of the mean number of clusters, $K(p)$. This is in principle obtained by subtracting from $K(p)$ terms which are analytic in $(p - p_c)$. However, if $K(p)$ is analytic at p_c we cannot distinguish between the 'singular part' and the 'analytic background'. (In what follows we shall assume that $K(p)$ is analytic in $q = 1 - p$ in one dimension, as would be the case if $2 - \alpha = L(L + 1)/2$, and that $2 - \alpha$ is defined by (2) with $K_s(p) = K(p)$. If α were not an integer hyperscaling would clearly fail.)

For systems in which $p_c = 1$ and $K(p)$ is analytic, the exponent $2 - \alpha$ is just the power of q in the leading term of the high density expansion of $K(p)$. Thus if $K_H(q)$ is the high density expansion of $K(p)$ about $q = 0$ we need only find the first term in this expansion. To this end we established the following results.

We define a *simple cut set* (Ω) of a connected graph G to be a collection of bonds, the removal of which divides G into exactly two connected subgraphs, both of which contain at least one bond, whereas the removal of any proper subset of Ω results in a single component graph.

Now suppose that G is a connected graph of N bonds, that G has $n_l > 0$ distinct cut sets of size l , all of which are simple, and that G has no cut sets of size less than l . We then have the following result.

Lemma. If $K_H(q)$ is the mean number of clusters per bond on G at bond density $p = 1 - q$ then

$$K_H(q) = 1/N + (n_l/N)q^l + O(q^{l+1}). \quad (3)$$

To prove this let C_k^b be the total number of distinct configurations on G with b bonds and k components. We then have

$$\sum_{k=0}^b C_k^b = \binom{N}{b}$$

$$C_1^b = \binom{N}{b}, \quad N - l < b \leq N$$

$$C_1^l = \binom{N}{l} - n_l$$

$$C_2^l = n_l$$

The last three equalities hold since there are no cut sets of size less than l , and n_l simple cut sets of size l .

Since each distinct configuration with a fixed number of bonds is equally likely, given the total number of bonds, we have

$$\mathcal{N}_b \equiv \langle \text{number of clusters} \mid b \text{ bonds} \rangle = N^{-1} \sum_{k=0}^b k C_k^b / \binom{N}{b}$$

where the factor N^{-1} accounts for normalisation per bond and $\langle \dots \rangle$ denotes expectation

over an ensemble of systems with b occupied bonds. To find $K_H(q)$ we note that

$$K_H(q) = \sum_{b=0}^N \mathcal{N}_b \Pr\{b \text{ bonds}|q\}$$

where $\Pr\{b \text{ bonds}|q\} = \binom{N}{b} p^b q^{N-b}$ is the probability of occupying b bonds given a probability p of occupying a single bond. We thus have

$$\begin{aligned} K_H(q) &= N^{-1} \sum_{b=0}^N \sum_{k=0}^N k C_k^b p^b q^{N-b} \\ &= N^{-1} \sum_{v=0}^N \sum_{k=0}^{N-v} k C_k^{N-v} q^v p^{N-v} \end{aligned} \tag{4}$$

where the sum over v corresponds to a sum over v vacant bonds. Using (4) and rearranging terms this becomes

$$\begin{aligned} K_H(q) &= N^{-1} \left(\sum_{v=0}^l \binom{N}{v} q^v (1-q)^{N-v} + n_l q^l (1-q)^{N-l} + \sum_{v=l+1}^N \sum_{k=0}^{N-v} k C_k^{N-v} q^v (1-q)^{N-v} \right) \\ &= N^{-1} \left\{ 1 + n_l q^l + q^{l+1} \left[n_l \sum_{i=1}^{N-l} \binom{N-l}{i} (-q)^{i-1} \right. \right. \\ &\quad \left. \left. - \sum_{v=l+1}^N q^{v-l-1} (1-q)^{N-v} \left(\binom{N}{v} - \sum_{k=0}^{N-v} k C_k^{N-v} \right) \right] \right\}. \end{aligned} \tag{5}$$

Noting that the term in square brackets is $O(q^0)$ equation (3) follows.

We can apply these results to an infinite graph G provided that there exists a sequence of graphs $\{G^N\}$ and a number $a_l > 0$ such that $n_l/N = a_l + O(N^{-1})$ as N tends to infinity. Then provided the series on the right-hand side of (5) converges when written as a power series in q , the leading term in $K_H(q)$ on G is simply $a_l q^l$.

For the usual lattices of dimension two or higher the above result is easily obtained from an expansion of the first few terms of the 'perimeter' series for $K(p)$, namely

$$K(p) = \sum_{b=1}^{\infty} p^b D_b(q)$$

where $D_b(q)$ is the 'perimeter' polynomial for clusters of b bonds on L . In this case one may obtain the first-order term in $K_H(q)$ directly to find that l is just the size of the smallest perimeter on L . This can be done because in general all finite cut sets on L are perimeters of small clusters. In one dimension this is not the case and the above lemma simply points out that the leading term of $K_H(q)$ is determined by the smallest cut sets, whether perimeters of small clusters or not. An analogous version of the above lemma also holds for site problems. For the site problem on I^L the smallest cut set is a sequence of L consecutive sites, the size of which is less than the size ($2L$) of the smallest cluster perimeter.

For the bond problem on I^L we note that the size $l_1(L)$ of the perimeter of a bond is given by

$$l_1(L) = 4L - 2$$

while the smallest number $l_2(L)$ of bonds which have to be removed to create two infinite graphs is

$$l_2(L) = 1 + 2 + \dots + L = L(L + 1)/2.$$

For $L \leq 6$, $l_2(L) < l_1(L)$, and for $L \geq 7$, $l_2(L) > l_1(L)$. From the lemma

$$2 - \alpha = \min(l_1(L), l_2(L)).$$

Hence, for $L \leq 6$, $\nu = 2 - \alpha$ but $\nu \neq 2 - \alpha$ for $L \geq 7$.

To understand this result one notices that the connectedness length (the analogue of the correlation length) is determined by $l_2(L)$ since isolating a bond does not necessarily destroy long-range connectivity. Hence ν is always determined by $l_2(L)$. On the other hand, since isolating a bond changes the number of clusters, α is determined by the minimum of l_1 and l_2 . Thus, for $L \leq 6$ there is a single important length scale in the critical regime, while for $L > 6$ there exist at least two important length scales and it is this feature which leads to the failure of hyperscaling.

In summary, long-range bond percolation in one dimension exhibits a crossover from a regime in which hyperscaling is obeyed, to a regime in which hyperscaling, and consequently weak universality, fail. The reason for the failure is the appearance of the perimeter set as the dominating cluster forming process in the critical regime. No crossover exists in the site problem on I^L , although it has been noted that the Ising problem with n -neighbour interactions exhibits a similar crossover when it becomes energetically favourable to flip a single spin (in zero field) as opposed to flipping a whole chain segment (Stauffer 1979).

References

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